Perturbation Theory for an Exactly Soluble Spinor Model in Interaction with Its Electromagnetic Field

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A nonlinear spinor field is studied interacting at rest with the static radial electromagnetic potential, A_0. The results of perturbation theory fully agree with the exact results previously obtained.

I. INTRODUCTION

The possibility of relying on a perturbative approach is usually an important step in any physical theory. This is particularly the case in dealing with classical models such as a charged spinor field, since the noninteracting spinor field is separable at rest, but the general interaction model is not. Perturbation theory affords the advantage of reducing the solution of the general problem to that of the bare case plus a simple iterative procedure. The noninteracting spinor field must have a nonlinear self-coupling in order to obtain solutions that are acceptable at the classical level, but this does not complicate the perturbative approach since all the process is linear once the nonlinear first step (the noninteracting field) has been solved.

In a previous paper[1] we obtained exact numerical solutions, at rest, of the problem of the interaction of a nonlinear spinor field and an electromagnetic potential A_μ such that $A_0 = A(r)$, $A_k = 0$ for $k = 1, 2, 3$. The details of the perturbative approach were left for the present publication. It was felt that coincidence of the exact and the perturbative results in this particularly simple model was important to warrant the use of perturbative methods in the general case in which an exact treatment is not available. The results which we present here would seem to fully justify the accuracy of perturbation methods. At the same time, they provide an important check for the exact numerical solutions formerly obtained for which no rigorous mathematical proofs could be found.

II. PERTURBATIVE EXPANSION AND FIRST-ORDER THEORY

It has been previously shown that the interaction of a nonlinear Dirac field with the electromagnetic field of which it is the source, i.e., the system derived from the Lagrangian density

$$L = \tfrac{1}{2}i[\bar\psi\gamma^\mu\partial_\mu\psi - (\partial_\mu\bar\psi)\gamma^\mu\psi] - eA_\mu\bar\psi\gamma^\mu\psi - \tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} - m\bar\psi\psi + \lambda(\bar\psi\psi)^2 \tag{1}$$

can be studied at rest with wave functions of the form

$$\psi = e^{-i\omega t}\begin{pmatrix} g\begin{pmatrix}1\\0\end{pmatrix} \\ if\begin{pmatrix}\cos\theta \\ e^{i\varphi}\sin\theta\end{pmatrix}\end{pmatrix}, \tag{2}$$

where g and f are real radial functions.

In the simplified case in which only $A_0 = A(r)$ is nonzero, and $A_1 = A_2 = A_3 = 0$, the Euler-Lagrange equations coming from (1) are

$$\begin{aligned} &g' + f[m + \omega - eA + 2\lambda(f^2 - g^2)] = 0, \\ &f' + \frac{2}{r}f + g[m - \omega + eA + 2\lambda(f^2 - g^2)] = 0, \\ &A'' + \frac{2}{r}A' + e(f^2 + g^2) = 0\ . \end{aligned} \tag{3}$$

If we define dimensionless fields $F, G, \mathcal{A}$, in the form

$$g = \left(\frac{m}{2\lambda}\right)^{1/2} G, \qquad A = \frac{m}{e}\mathcal{A},$$

$$f = \left(\frac{m}{2\lambda}\right)^{1/2} F, \qquad r = \frac{\rho}{m},$$

and

$$\Lambda = \frac{\omega}{m},$$

we have

$$\begin{aligned} &G' + F(1 + \Lambda - \mathcal{A} + F^2 - G^2) = 0, \\ &F' + \frac{2}{\rho}F + G(1 - \Lambda + \mathcal{A} + F^2 - G^2) = 0, \\ &\mathcal{A}'' + \frac{2}{\rho}\mathcal{A}' + \epsilon(F^2 + G^2) = 0, \end{aligned} \tag{4}$$

where

$$\epsilon=\frac{e^2}{2\lambda m^2}\,.$$

For this system localized particlelike solutions were found up to values of $\epsilon \lesssim 0.6$, having an exponential decay and a Λ-dependent total energy with a minimum given by the empirical formula[1]

$$E^{\min}-E_0^{\min}=\sigma(\Lambda^{\min}-\Lambda_0^{\min})\,.$$

We can now expand the fields F, G, $\mathfrak{A}$ and the conconstant Λ in the form

$$\begin{aligned}
&\mathfrak{A}=\epsilon\mathfrak{A}_1+\epsilon^2\mathfrak{A}_2+\cdots,\\
&F=F_0+\epsilon F_1+\epsilon^2F_2+\cdots,\\
&G=G_0+\epsilon G_1+\epsilon^2G_2+\cdots,\\
&\Lambda=\Lambda_0+\epsilon\Lambda_1+\epsilon^2\Lambda_2+\cdots.
\end{aligned}\tag{5}$$

Substitution of (5) in (4) gives respectively in zero and first order in ϵ the equations

$$\begin{aligned}
&G_0'+F_0(1+\Lambda_0+F_0^2-G_0^2)=0,\\
&F_0'+\frac{2}{\rho}F_0+G_0(1-\Lambda_0+F_0^2-G_0^2)=0,
\end{aligned}\tag{6}$$

$$G_1'+F_1(1+\Lambda_0+3F_0^2-G_0^2)-2G_1F_0G_0+F_0(\Lambda_1-\mathfrak{A}_1)=0,$$

$$\begin{aligned}
&F_1'+\frac{2}{\rho}F_1+G_1(1-\Lambda_0+F_0^2-3G_0^2)\\
&\qquad+2F_1F_0G_0-G_0(\Lambda_1-\mathfrak{A}_1)=0,\\
&\mathfrak{A}_1''+\frac{2}{\rho}\mathfrak{A}_1'+F_0^2+G_0^2=0\,.
\end{aligned}\tag{7}$$

Higher-order equations are easily obtained.

In first order we get for the energy

$$\mathcal{E}_1=\frac{2\pi}{\lambda m}E_1,$$

where

$$\begin{aligned}
E_1=E_0+\epsilon\int\rho^2d\rho[&(\Lambda_1-\mathfrak{A}_1)(F_0^2+G_0^2)\\
&+2(F_0F_1-G_0G_1)(F_0^2-G_0^2)\\
&+2\Lambda_0(F_0F_1+G_0G_1)+\tfrac{1}{2}\mathfrak{A}_1'^2].
\end{aligned}\tag{8}$$

We will now explain in some detail the peculiarities inherent to the perturbative solution, which are more conveniently described in first order. In the first place, it may seem superfluous to expand Λ in the form shown in (5). The need for doing so is quite compelling, since it is the only way to preserve gauge invariance in all orders of perturbation theory. In this particular model it means that at each order n, the equations remain unchanged under the transformations $\Lambda_n\to\Lambda_n+\Omega_n$, $\mathfrak{A}_n\to\mathfrak{A}_n+\Omega_n$, where Ω_n are arbitrary constants. We will always choose for $\mathfrak{A}_n$ the natural gauge in which $\mathfrak{A}_n\to 0$ as $\rho\to\infty$.

The expansion of Λ might appear however to confuse the situation, since an infinity of new parameters Λ_n takes the place of the only parameter Λ, which appeared in the exact theory. The answer is of course that all the Λ_n can and must be fixed at each order n, as functions of Λ_0, the value of Λ for the noninteracting case. This is the same as saying that Λ_0 must be shifted or "renormalized" in each order because of the existence of the electromagnetic field. One thus sees from a different point of view the need for expanding Λ as in (5). Had we not made this provision, and fixed $\Lambda=\Lambda_0$ in zero order, and had calculated afterwards F_1, G_1 in first order (with Λ_1 implicitly equal to zero) it will soon be seen that a wrong result would have obtained.

Our next aim is to obtain the dependence of the energy E_1 (up to and including first order in ϵ) with Λ_0 once Λ_1 is fixed. This involves finding first how the energy E_1 varies for each Λ_0 when we make Λ_1 adopt arbitrary values. It is easy to show that the variation is linear. In fact, we may write the general solution of Eq. (7) in the form

$$\begin{aligned}
&F_1=F_{11}(\rho)+\Lambda_1F_{12}(\rho),\\
&G_1=G_{11}(\rho)+\Lambda_1G_{12}(\rho)\,.
\end{aligned}\tag{9}$$

Since Eq. (7) are linear in F_1 and G_1, one can solve separately for the terms of order 0 or 1 in Λ_1 and thus obtain the unique functions F_{11}, F_{12}, G_{11}, G_{12}, which satisfy the boundary conditions (regular at the origin and vanishing at infinity). F_1 and G_1 will obviously satisfy these conditions if F_{11}, F_{12}, G_{11}, and G_{12} do so. They are also uniquely determined for each Λ_1. If we substitute the expressions for F_1, G_1 in the energy formula (8), we immediately see that E_1 depends linearly on Λ_1 for each value of Λ_0.

At this point it is useful to visualize the situation as in Fig. 1, where the numerical results are plotted. For several values of Λ_0 the variation of E_1 with Λ_1 appears in the form of straight lines.

Now, it is clear that first-order perturbation theory should give a continuous variation of the energy with Λ_0. Otherwise it could not be expected to apply. Therefore, if we choose two values of Λ_0, say Λ_0 and $\Lambda_0+\Delta\Lambda_0$, which are very close, the perturbative values for the energy should also be very close. Since these values lie on the two straight lines corresponding to Λ_0 and $\Lambda_0+\Delta\Lambda_0$, the only value of E_1 which can be expected to vary continuously as $\Delta\Lambda_0\to 0$ is the limiting intersecting point of both lines, i.e., *the point at which the*

straight lines are tangent to their envelope. One point is thus fixed in each tangent, and therefore a value of $\Lambda = \Lambda_0 + \epsilon\Lambda_1$ for each Λ_0. In other words, Λ_1 is thus fixed as a function of Λ_0.

In Fig. 1 the numerical results can be seen to substantiate our conclusions. The envelope of the set of straight lines, representing E_1, clearly differs from the exact results E by amounts of second or higher orders.

III. PERTURBATION THEORY AT ANY ORDER

What has been done in first order can be followed closely in higher orders.

Once the parameters $\Lambda_1, \ldots, \Lambda_{n-1}$ are found as functions of Λ_0, the nth-order wave functions F_n, G_n depend linearly on Λ_n. Then E_n, the energy up to and including nth order, also depends linearly on Λ, since up to this order

$$\Lambda = \Lambda_0 + \epsilon\Lambda_1 + \cdots + \epsilon^{n-1}\Lambda_{n-1} + \epsilon^n\Lambda_n \ . \tag{10}$$

Assuming analyticity in ϵ and Λ we can therefore write the following series expansion:

$$E_n(\epsilon, \Lambda(\epsilon)) = E_0(\Lambda_0) + (\Lambda - \Lambda_0)W + \sum_{k=1}^{n} \epsilon^k V_k \ . \tag{11}$$

W and V_k are functions of Λ_0. Explicitly

$$W(\Lambda_0) = \frac{\partial E_n(\epsilon, \Lambda(\epsilon))}{\partial \Lambda}\bigg|_{\epsilon = 0} \ . \tag{12}$$

The envelope of these lines will be, by an argument identical to the one made in the last section, the curve $E_n = E_n(\Lambda_0)$ describing the dependence of the energy on Λ_0 up to and including terms of nth order in ϵ. It is obtained by eliminating Λ between (11) and its derivative with respect to Λ_0, i.e.,

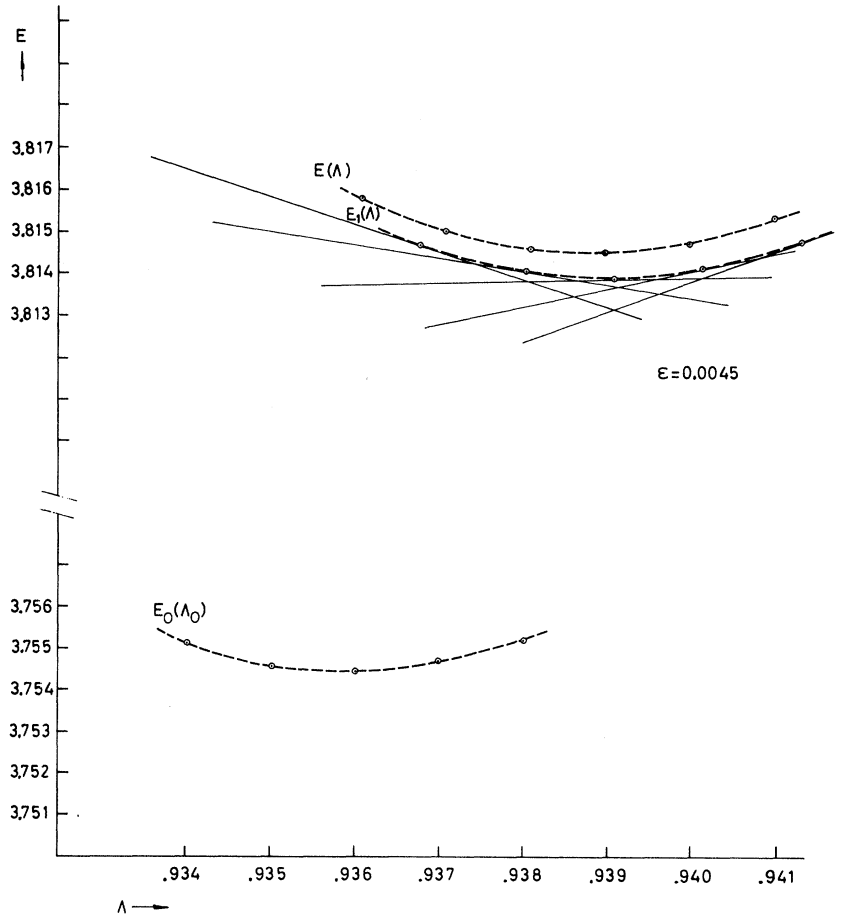

FIG. 1. Results of numerical calculations for the energy as a function of Λ. For the noninteracting system Λ coincides with Λ_0, and the lowest curve $E_0 = E_0(\Lambda_0)$ is obtained. In first order $\Lambda = \Lambda_0 + \epsilon\Lambda_1$ and E_1 varies along the straight lines for each value of Λ_0 as Λ_1 takes different values. The only meaningful value of E_1 for each of the lines is the one at which they are tangent to their envelope. We include for comparison the results of the exact theory, lying on the upper curve, $E(\Lambda)$.

$$0 = E_0' + \Lambda W' - W - \Lambda_0 W' + \sum_{k=1}^{n} \epsilon^k V_k' . \tag{13}$$

Having (12) in mind,

$$\Lambda = \Lambda_0 - \frac{1}{W'} \sum_{k=1}^{n} \epsilon^k V_k'$$

which determines the values of Λ_n as

$$\Lambda_n = -\frac{1}{W'} V_n' . \tag{14}$$

As $n \to \infty$ we have thus a value of Λ given by (10) which determines $\Lambda(\epsilon)$ as a function of Λ_0 and also a function $E_n(\epsilon, \Lambda(\epsilon))$. Both Λ and E_n should coincide in the limit (within the domain of analyticity of E in ϵ and Λ) with the values Λ and E obtained numerically in the exact theory.

An interesting point which follows immediately from Eq. (11) is that for each Λ_0 the corresponding straight lines given by (11) are at any order parallel to one another and also to the tangent to $E_0 = E_0(\Lambda_0)$ at Λ_0. The exact theory shares of course this same property.

As a very useful consequence the minima at any order (and therefore also the minimum of the exact theory) all correspond to $\Lambda_0^{\min}$, the Λ_0 value giving the minimum for the noninteracting case.

In practice, therefore, when applying perturbation theory at any order, one needs only calculate the functions F_n, G_n, corresponding to $\Lambda_0^{\min}$ since these will also give the minimum for E_n. Working in first order, if what one looks for is just the value of $E_1^{\min}$, it is not even necessary to determine Λ_1, since $E_1^{\min}$ is independent of this parameter. $E_1^{\min}$, which is the physically relevant value for the energy of the system up to first order, is therefore quite easy to obtain once $\Lambda_0^{\min}$ has been determined.

We are also in a position to discuss the empirically found fact that all the exact minima of the energy for the interacting system were lying approximately on a straight line. This was described by the tentative relation

$$E^{\min} - E_0^{\min} = \sigma(\Lambda^{\min} - \Lambda_0^{\min}).$$

If $E^{\min}$ is substituted by $E_1^{\min}$, the relation is exact, and one easily finds

$$\sigma = E_0' - \frac{V_1}{V_1'} E_0'' . \tag{15}$$

In higher orders, however, the relation does not hold.

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[1]M. Soler, preceding paper, Phys. Rev. D 8, 3424 (1973).